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# A real Riemann-Hurwitz theorem

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**Abstract.** We prove a real version of the Riemann-Hurwitz theorem and apply it to solve a problem of enumerative geometry in the real case: the number of plane projective curves tangent to a line and passing through the appropriate number of points.

**Keywords:** Ramificaton points, plane curves.

## 1. Introduction

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and consider the following enumerative problem: let $n$ be a positive integer and set $N_n = \frac{n(n+3)}{2}$, which is the dimension of the projective space $\mathbb{P}_{n,\mathbb{K}}$ of curves of degree $n$ in the projective plane $\mathbb{P}^2_{\mathbb{K}}$. Then, given $k$ lines $\ell_1, \ldots, \ell_k$ and $N_n - k$ points $P_{k+1} \ldots, P_{N_n}$ in general position in $\mathbb{P}^2_{\mathbb{K}}$, there is a finite number of curves of degree $n$ which are tangent to $\ell_1, \ldots, \ell_k$ and pass through $P_{k+1}, \ldots, P_{N_n}$.

If we are working over the complex numbers, then the number $s_k$ of such curves does not depend on the choice of the generic $\ell$'s and $P$'s; as we shall see in § 1, if $k$ is small enough so that there is no curve of degree $n$ with a double component through the $P$'s (i.e. $k < 2n - 1$), then it follows from the Riemann-Hurwitz formula that $s_k = (2(n-1))^k$. This is well known and easy to prove; it should provide support for what follows.

The aim of this paper is to show that in the real case, for $k = 1$, for any $r$ such that $0 \leq r \leq n - 1$ there exist configurations of real points $P_2, \ldots, P_{N_n}$ and a line $\ell$ such that there are $2(n-1) - 2r$ real curves of degreee $n$ passing trough $P_2, \ldots, P_{N_n}$ and tangent to $\ell$ at a real point. Our main ingredient is a real version of the Riemann-Hurwitz theorem. Such questions have raised interest in recent years (see for example [**4**]).

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## 2. Reduction to the Riemann-Hurwitz theorem in the complex case

Let $1 \leq N \leq N_n$. For $P_1, \ldots, P_N \in \mathbb{P}^2_{\mathbb{K}}$. Set

$$\mathcal{L}_{P_1,\ldots,P_N} = \left\{[f] \in \mathbb{P}_{n,\mathbb{K}} \mid f(P_1) = \cdots = f(P_N) = 0\right\}.$$

One expects that if $P_1, \ldots, P_N$ are in general position, then $\mathcal{L}_{P_1,\ldots,P_N}$ has codimension $N$ in $\mathbb{P}_{n,\mathbb{K}}$. Proposition 1 below establishes that it makes sense to speak of a "generic choice" of a line and points for our enumerative problem.

**Proposition 1.** *For all $2 \leq N \leq N_n$ the set*

$$\Omega = \left\{(\ell, P_2, \ldots, P_N) \in \check{\mathbb{P}}^2_{\mathbb{K}} \times \left(\mathbb{P}^2_{\mathbb{K}}\right)^{N-1} \mid \forall P \in \ell \ \dim(\mathcal{L}_{P,P_2,\ldots,P_N}) = N_n - N\right\}$$

*is non-empty and Zariski-open in $\check{\mathbb{P}}^2_{\mathbb{K}} \times \left(\mathbb{P}^2_{\mathbb{K}}\right)^{N-1}$.*

**Proof.** Clearly, the set $\Omega$ is Zariski-open. In § 3, Proposition 5, we will show explicit points $Q_2, \ldots, Q_{N_n}$ and a line $\ell$ such that $\dim(\mathcal{L}_{Q,Q_2,\ldots,Q_{N_n}}) = 0$, for all $Q \in \ell$. □

So let $(\ell, P_2, \ldots, P_{N_n}) \in \Omega$, and define

$$Z = \left\{(P, [f]) \in \mathbb{P}^2_{\mathbb{K}} \times \mathbb{P}_{n,\mathbb{K}} \mid P \in \ell\,,\ f(P) = f(P_2) = \cdots = f(P_{N_n}) = 0\right\}.$$

We have the two natural projections

$$\begin{array}{ccc} Z & \xrightarrow{\pi_1} & \ell \\ {\scriptstyle \pi_2}\downarrow & & \\ \mathcal{L}_{P_2,\ldots,P_{N_n}} & & \end{array}$$

and $\pi_1$ is an isomorphism because $(\ell, P_2, \ldots, P_{N_n}) \in \Omega$. Therefore, we can define

$$\phi\colon \ell \to \mathcal{L}_{P_2,\ldots,P_{N_n}}, \quad \phi(P) = \pi_2(\pi_1^{-1}(P)) = \text{the unique curve of degree } n \text{ through } P,\ P_2, \ldots, P_{N_n}.$$

Clearly, $P$ is a ramification point of $\phi$ if and only if the curve $\phi(P)$ is tangent to $\ell$ at the point $P$. Assume now that $\mathbb{K} = \mathbb{C}$. According to the Riemann–Hurwitz theorem the number of ramification points of $\phi$ equals

$$\text{degree of } \phi \times \text{Euler characteristic of } \mathcal{L}_{P_2,\ldots,P_{N_n}} - \text{Euler characteristic of } \ell = 2n - 2 = 2(n-1),$$

and so this is also the number of curves of degree $n$ through $P_2, \ldots, P_{N_n}$ and tangent to $\ell$.

This also shows that, for $2 \leq N \leq N_n$, the subvariety of $\mathbb{P}^2_{\mathbb{K}}$,

$$\mathcal{L}_{\ell, P_2, \ldots, P_N} = \{([f] \in \mathbb{P}^2_{\mathbb{K}} \mid \text{the curve } f(x) = 0 \text{ is tangent to } \ell \text{ and } f(P_2) = \cdots = f(P_N) = 0\},$$

has degree $2(n-1)$. Then one is tempted to argue that, given $k$ lines $\ell_1, \ldots, \ell_k$ and $N_n - k$ points $P_{k+1}, \ldots, P_{N_n}$ in general position, the set

$$\mathcal{L}_{\ell_1, \ldots, \ell_k, P_{k+1}, \ldots, P_{N_n}} = \{([f] \in \mathbb{P}^2_{\mathbb{K}} \mid f \text{ is tangent to } \ell_1, \ldots, \ell_k \text{ and } f(P_{k+1}) = \cdots = f(P_{N_n}) = 0\}$$

consists of $(2(n-1))^k$ points, since

$$\mathcal{L}_{\ell_1, \ldots, \ell_k, P_{k+1}, \ldots, P_{N_n}} = \bigcap_{i=1,\ldots,k} \mathcal{L}_{\ell_i, P_{k+1}, \ldots, P_{N_n}}. \qquad (\#)$$

However, this argument works only for $k < 2n - 1$, because otherwise there is a curve $g$ of degree $n$ through $P_{k+1}, \ldots, P_{N_n}$ having a double line as component, and so $g$ is tangent to any line in the plane (see [**2**]). In fact, this shows that in (#) there might be a residual intersection to account for. For example, in the case of conics, there are 2 conics through 4 points and tangent to one line, there are $4 = 2^2$ conics through 3 points and tangent to 2 lines, but the number of conics through 2 points and tangent to 3 lines is not $2^3 = 8$: it is 4, the same as the number of conics through 3 points tangent to three lines, as one can see by resorting to the dual conics. In [**1**] the case of cubics and in [**5**] the case of quartics are considered, over the complex numbers.

It would be desirable to know in addition that for a generic choice of $\ell, P_2, \ldots, P_{N_n}$, the curves through $P_2, \ldots, P_{N_n}$ tangent to $\ell$ are simply tangent to $\ell$ (no double tangents, no inflection points, nor worse), which amounts to say that the map $\phi$ has only ordinary ramification points, with distinct values. This can be done by studying more closely the family of maps $\phi_{\ell, P_2, \ldots, P_{N_n}}$, where now $\ell, P_2, \ldots, P_{N_n}$ are movable parameters. In § 3 we will satisfy this desire by exhibiting $\ell, P_2, \ldots, P_{N_n}$ such that the corresponding $\phi$ has $2(n-1)$ distinct ramification points, which must therefore be ordinary ramification points, with distinct values.

## 3. A real version of the Riemann-Hurwitz theorem

Let

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 1 = 0\}$$

denote the topological circle. We shall say that the (disjoint) subsets $F_1, \ldots, F_h$ of $S^1$ are non interlaced if there exist open, disjoint subintervals $I_i$, $i = 1, \ldots, h$ of $S^1$ such that $F_i \subset I_i$, $i = 1, \ldots, h$.

**Proposition 2.** *Let $f: S^1 \to S^1$ be a $C^1$ map whose fibers are all finite. If there are distinct $y_1, \ldots, y_h \in S^1$ such that $f^{-1}(y_1), \ldots, f^{-1}(y_h)$ are not interlaced, denoting by $n_i$ the number of points in $f^{-1}(y_i)$, there are at least*

$$\sum_{i=1}^{h} n_i - h$$

*points where the derivative of $f$ vanishes. More precisely, if $f^{-1}(y_i) \subset I_i$, where the $I_i$'s are disjoint intervals, and*

$$f^{-1}(y_i) = \left\{a_1^i, \ldots, a_{n_i}^i\right\} \quad \text{with} \quad a_1^i < \cdots < a_{n_i}^i$$

*for some ordering of the interval $I_i$, then there is at least one ramification point between $a_j^i$ and $a_{j+1}^i$, for $j = 1, \ldots, n_i - 1$.*

**Proof.** $f$ maps $I_i$ into $S^1 \setminus \{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_h\}$, which is diffeomorphic to a subset of $\mathbb{R}$. Then it follows from Rolle's theorem that for all $j \in \{1, \ldots, n_i - 1\}$ there exists $c$, $a_j^i < c < a_{j+1}^i$, with $f'(c) = 0$. □

**Corollary.** *Let $\Theta = [F : G]\colon \mathbb{PR}^1 \to \mathbb{PR}^1$, where $F(x, t)$ and $G(x, t)$ are homogeneous polynomials of degree $n$ without common non-trivial zeroes. Let $N = [1 : 0]$ and $S = [0 : 1]$, and assume that $\Theta^{-1}(N) = \{P_1, \ldots, P_n\}$, $\Theta^{-1}(S) = \{Q_1, \ldots, Q_n\}$ and that these 2 sets are not interlaced. Then $\Theta$ has exactly $2(n-1)$ ramification points. More precisely, after perhaps renumbering the $P_i$'s and the $Q_i$'s, we can choose $Z \in \mathbb{PR}^1 \setminus \{P_1, \ldots, P_n, Q_1, \ldots, Q_n\}$ and an order of $\mathbb{PR}^1 \setminus Z$ such that $P_1 < \cdots < P_n < Q_1 < \cdots < Q_n$. Then for $i = 1, \ldots, n-1$ there exist $A_i, B_i$, $P_i < A_i < P_{i+1}$, $Q_i < B_i < Q_{i+1}$, such that $\Theta'(A_i) = \Theta'(B_i) = 0$.*

**Proof.** Proposition 2 ensures the existence of ramification points as stated. There cannot be more, for otherwise $\Theta' \equiv 0$ and $\Theta$ would be constant. □

On the opposite end, we have:

**Proposition 3.** *Let $\Theta = [F : G]$, $F(x,t)$ and $G(x,t)$ homogeneous of degree $n$ without non-trivial common zeroes, be such that $\Theta^{-1}(N) = \{P_1, \dots, P_n\}$, $\Theta^{-1}(S) = \{Q_1, \dots, Q_n\}$, and assume that each component of $\mathbb{PR}^1 \setminus \{Q_1, \dots, Q_n\}$ contains exactly one of the $P_i$'s (and hence each component of $\mathbb{PR}^1 \setminus \{P_1, \dots, P_n\}$ contains exactly one of the $Q_i$'s). Then $\Theta$ has no real ramification point.*

**Proof.** $\mathbb{PR}^1 \setminus \{N, S\} = I \cup J$, where $I$ and $J$ are disjoint intervals. For $y \in I$ near $N$, $\Theta^{-1}(y)$ has still $n$ distincts preimages, one in each component of $\mathbb{PR}^1 \setminus \{Q_1, \dots, Q_n\}$; therefore, as we move $y \in I$ towards $S$, it cannot happen that 2 distinct preimages come together. If $y_0$ is the first critical value that we encounter, then, since $\Theta$ is proper, $\Theta^{-1}(y_0)$ also has one preimage in each of the $n$ components of $\mathbb{PR}^1 \setminus \{Q_1, \dots, Q_n\}$, and one of these preimages has multiplicity, a contradiction. □

In order to produce a morphism $\Theta\colon \mathbb{PR}^1 \to \mathbb{PR}^1$ with any number of ramification points of the form $2(n-1) - 2r$, $0 \le r \le n-1$, we want to use the old trick that consists in joining a morphism $\Theta_{\max}$ with $2(n-1)$ ramification points to a morphism $\Theta_{\min}$ with no ramification points, through a "generic" path in the space of morphisms. Such a generic path will encounter 2 kinds of catastrophes: $F$ and $G$ can acquire a non-trivial common zero, or $\Theta$ can have a ramification point of order 2. One hopes that when crossing a catastrophic situation, the number of ramification points changes by $\pm 2$, which ensures that all values $2(n-1) - 2r$, $0 \le r \le n-1$ are taken when moving from $\Theta_{\max}$ to $\Theta_{\min}$. The details are provided by Proposition 4.

We may assume without loss of generality that $F(1,0) \neq 0$, $G(1,0) \neq 0$ and that $[1:0]$ is not a critical point of $\Theta$. Then the morphism $\Theta$ can be written in affine coordinates as follows:

$$\theta(x) = \frac{f(x)}{g(x)}, \ f(x) = x^n + a_1 x^{n-1} + \dots + a_n, \ g(x) = x^n + b_1 x^{n-1} + \dots + b_n.$$

Let $\varphi(x, f, g) = f'(x) \cdot g(x) - f(x) \cdot g'(x)$, so that $\theta'(x) = \varphi(x)/g(x)^2$. Note that $f'(x) \cdot g(x)$ and $f(x) \cdot g'(x)$ have same leading coefficient, therefore $\varphi(x)$ has degree $2(n-1)$. The zeroes of $\varphi$ are exactly the ramification points of $\theta$ when $f$ and $g$ have no common factor, and vanishes for $x = \alpha$ if $f$ and $g$ have $x - \alpha$ as a common factor.

We denote by $A_n$ the space of polynomials of degree $\le n$ in $x$ with real coefficients and leading coefficient 1; thus $f, g \in A_n$.

**Proposition 4.** *Assume $n \geq 2$. There exists an algebraic subset $Z \subset A_n \times A_n$ of codimension at least 2 such that if $(f, g) \notin Z$, then*

$$\varphi(x, f, g) = \frac{\partial \varphi}{\partial x}(x, f, g) = 0 \quad \textit{implies that} \quad \frac{\partial^2 \varphi}{\partial x^2}(x, f, g) \neq 0.$$

**Proof.** First let $n = 2$, $f = x^2 + ax + b$, $g = x^2 + cx + d$. Then

$$\begin{aligned}\varphi(x, f, g) &= x^2(a - c) + 2x(b - d) + bc - ad,\\ \frac{\partial \varphi}{\partial x}(x, f, g) &= 2x(a - c) + (b - d),\\ \frac{\partial^2 \varphi}{\partial x^2}(x, f, g) &= 2(a - c),\end{aligned}$$

so that $\frac{\partial \varphi}{\partial x}(x, f, g) = \frac{\partial^2 \varphi}{\partial x^2}(x, f, g) = 0$ implies that $f = g$. If, for $n \geq 2$, we take $Z_1$ to be the Zariski closure of the set

$$\{(f, g) \in A_n \times A_n \mid f \text{ and } g \text{ have at least 2 common roots}\},$$

then $Z_1$ has codimension at least 2 in $A_n \times A_n$, and so for $n = 2$ we can take $Z = Z_1$.

Let now $n \geq 3$ and consider

$$W = \left\{(x, f, g) \in \mathbb{R} \times A_n \times A_n \;\middle|\; \varphi(x, f, g) = \frac{\partial \varphi}{\partial x}(x, f, g) = \frac{\partial^2 \varphi}{\partial x^2}(x, f, g) = 0\right\}.$$

If $(x, f, g) \in W$ and $g(x) \neq 0$, we take the derivatives of the 3 equations defining $W$ in the direction of a vector of the form $(0, \overline{f}, 0) \in \mathbb{R} \times A_n \times A_n$ and obtain:

a) $\varphi(x, f, g) = f'(x) \cdot g(x) - f(x) \cdot g'(x) = 0 \quad \longrightarrow \quad \overline{f}'(x) \cdot g(x) - \overline{f}(x) \cdot g'(x) = 0$

b) $\frac{\partial \varphi}{\partial x}(x, f, g) = f''(x) \cdot g(x) - f(x) \cdot g''(x) = 0$
$\longrightarrow \quad \overline{f}''(x) \cdot g(x) - \overline{f}(x) \cdot g''(x) = 0$

c)

$$\frac{\partial^2 \varphi}{\partial x^2}(x, f, g) = f'''(x) \cdot g(x) + f''(x) g'(x) - f'(x) \cdot g''(x) - f(x) \cdot g'''(x) = 0$$

$$\longrightarrow \quad \overline{f}'''(x) \cdot g(x) + \overline{f}''(x) \cdot g'(x) - \overline{f}'(x) \cdot g''(x) - \overline{f}(x) \cdot g'''(x) = 0$$

Since $\overline{f}$ is of degree at least 3, we can find $\overline{f}$ such that:

-a) $\overline{f}(x) = 0, \overline{f}'(x) \neq 0$, which does not satisfy a) above

a-b) $\overline{f}(x) = \overline{f}'(x) = 0, \overline{f}''(x) \neq 0$, which satisfies a) but not b)

a+b-c) $\overline{f}(x) = \overline{f}'(x) = \overline{f}''(x) = 0, \overline{f}'''(x) \neq 0$, which satisfies a) and b) but not c).

It follows that the three equations defining $W$ are of maximal rank at $(x, f, g)$. If $g(x) = 0$ and $f(x) \neq 0$, the above argument can be repeated with the roles of $f$ and $g$ exchanged. We set

$$W_1 = \{(x, f, g) \in W \mid f \text{ and } g \text{ have no real common root}\}$$

and let $Z_2$ be the Zariski closure of $p(W_1)$, with $p\colon \mathbb{R} \times A_n \times A_n \to A_n \times A_n$ the natural projection. Then, since $W_1$ has codimension 3 in $\mathbb{R} \times A_n \times A_n$, $Z_2$ has codimension at least 2 in $A_n \times A_n$.

Assume now that $f$ and $g$ have one real common root $\alpha$: $f(x) = (x-\alpha) f_1(x)$, $g(x) = (x - \alpha) g_1(x)$, but $f_1$ and $g_1$ without common root. Then

$$\begin{aligned}\varphi(x, f, g) &= (x-\alpha) f_1(x)(g_1(x) + (x-\alpha) g_1'(x)) \\ &\quad - (f_1(x) + (x-\alpha) f_1'(x))(x-\alpha) g_1(x) \\ &= (x-\alpha)^2 \varphi(x, f_1, g_1).\end{aligned}$$

We know by Equation a) of the first part of this proof that $\frac{\partial \varphi}{\partial f_1}(x, f_1, g_1) \neq 0$ or $\frac{\partial \varphi}{\partial g_1}(x, f_1, g_1) \neq 0$. Consequently, the Zariski closure $Z_3$ of the set of pairs $(f, g) \in A_n \times A_n$ having a common real root $\alpha$, for which $\varphi(\alpha, f_1, g_1) = 0$, has codimension at least 2 in $A_n \times A_n$. Since $\frac{\partial^2 \varphi}{\partial x^2}(\alpha, f, g) = 2\varphi(\alpha, f_1, g_1)$, the set $Z = Z_1 \cup Z_2 \cup Z_3$ will have the required properties. □

**Corollary 1.** *There exists an algebraic subset $Z_0 \subset \mathbb{R}^n \times \mathbb{R}^n$ of codimension at least 2 such that for $((\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)) \in \mathbb{R}^n \times \mathbb{R}^n \setminus Z_0$, if we set*

$$f(x) = \prod_{i=1,\dots,n} (x - \alpha_i), \quad g(x) = \prod_{i=1,\dots,n} (x - \beta_i) \text{ and } \varphi(x, \alpha, \beta) = \varphi(x, f, g),$$

*then*

$$\varphi(x, \alpha, \beta) = \frac{\partial \varphi}{\partial x}(x, \alpha, \beta) = 0 \quad \text{implies} \quad \frac{\partial^2 \varphi}{\partial x^2}(x, \alpha, \beta) \neq 0.$$

**Proof.** The map $\mathcal{N} : \mathbb{R}^n \to A_n$, $\mathcal{N}(\alpha_1, \ldots, \alpha_n) = \prod_{i=1,\ldots,n}(x - \alpha_i)$ has finite fibers, and so $Z_0 = (\mathcal{N} \times \mathcal{N})^{-1}(Z)$ has codimension 2 and has the required properties. □

**Corollary 2.** *For any $r$, $0 \leq r \leq n-1$, there exist $2n$ distinct real numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ such that the morphism*

$$\Theta(\alpha, \beta) = \left[ \prod_{i=1,\ldots,n} (x - \alpha_i t) : \prod_{i=1,\ldots,n} (x - \beta_i t) \right] : \mathbb{PR}^1 \to \mathbb{PR}^1$$

*has exactly $2(n-1) - 2r$ ordinary real ramification points.*

**Proof.** Take $\alpha_{\max} = \{-n, \ldots, -1\}$ and $\beta_{\max} = \{1, \ldots, n\}$, and take $\alpha_{\min} = \{2k-1,\ k = 1, \ldots, n\}$ and $\beta_{\min} = \{2k,\ k = 1, \ldots, n\}$. Then $\theta(\alpha_{\max}, \beta_{\max})$ has $2(n-1)$ real ramification points by Propostition 2, and $\theta(\alpha_{\min}, \beta_{\min})$ has no real ramification point according to Proposition 3. If we take a path that joins $(\alpha_{\max}, \beta_{\max})$ to $(\alpha_{\min}, \beta_{\min})$ and avoids the subset $Z_0$ of corollary 1, the corresponding $\varphi(x, \alpha, \beta)$ will have at worse a double root at $x \in \mathbb{R}$, so that the number of roots will change by $\pm 2$ along this path. □

## 4. Configurations with the maximal number of real solutions

Consider in $\mathbb{P}^2_{\mathbb{R}}$ two sets of $n$ distinct lines in general position, $\{\ell^1_0, \ldots, \ell^1_{n-1}\}$ and $\{\ell^2_0, \ldots, \ell^2_{n-1}\}$. Set $A_{i,j} = \ell^1_i \cap \ell^2_j$.

**Proposition 5.** *Let $\ell$ be a line not containing any of the $A_{i,j}$'s. Then the line $\ell$ and the points:*

$$A_{i,j} \text{ where } i + j \leq n \text{ and } 0 \leq i, j \leq n-1,$$

*are in general position in the sense that for all $P \in \ell$, there is exactly one curve of degree $n$ through $P$ and all the $A_{i,j}$.*

**Proof.** Set $I = \{(i, j) \mid i + j \leq n,\ 0 \leq i, j \leq n-1\}$. Then

$$I = \{(i, j) \mid i + j \leq n,\ 0 \leq i, j\} \setminus \{(n, 0), (0, n)\}.$$

Furthermore, the cardinality of $I$ is, as expected,

$$\frac{(n+1)(n+2)}{2} - 2 = \frac{n(n+3)}{2} - 1 = N_n - 1.$$

Assume first that $P \notin \cup_{i=0}^{n-1} \ell_i^1$. Let $[\phi_{i,j}] \in \check{\mathbb{P}}_{\mathbb{R}}^2$, $(i, j) \in I$, be the equation of a line such that $\phi_{i,j}(A_{i,j}) = 0$ and $\phi_{i,j}(A_{i',j'}) \neq 0$ for $(i, j) \neq (i', j')$ and such that $\phi_{i,j}(P) \neq 0$. Let $\psi_i$ be the equation of $\ell_i^1$.

The curves $\prod_{j=0}^{h} \phi_{0,j}$ for $0 \leq h \leq n-2$ go through $A_{0,0}, \ldots, A_{0,h}$, but no other of the $A_{i,j}$'s, nor $P$, and $\psi_0$ goes through $A_{0,0}, \ldots, A_{0,n-1}$, but no other of the $A_{i,j}$'s, nor $P$.

Next, for $0 \leq h \leq n-2$, $\psi_0 \cdot \prod_{j=0}^{h} \phi_{1,j}$ goes through $A_{0,0}, \ldots, A_{0,n-1}$, $A_{1,0}, \ldots, A_{1,h}$, but no other of the $A_{i,j}$'s, nor $P$, and $\psi_0 \cdot \psi_1$ goes through $A_{0,0}, \ldots, A_{0,n-1}, A_{1,0}, \ldots, A_{1,n-1}$, but no other of the $A_{i,j}$'s, nor $P$.

In general, for $0 \leq r \leq n-2$ and $0 \leq h \leq n-r-2$, $\psi_0 \cdot \cdots \cdot \psi_r \cdot \prod_{j=0}^{h} \phi_{r+1,j}$ will go through $A_{0,0}, \ldots, A_{r,n-r}, A_{r+1,1}, \ldots, A_{r+1,h}$, but no others, nor $P$, and $\psi_0 \cdot \cdots \cdot \psi_{r+1}$ through $A_{0,0}, \ldots, A_{r,n-r}, A_{r+1,1}, \ldots, A_{r+1,n-r-1}$, for $r+1 \leq n-1$, but no other of the $A_{i,j}$'s, nor $P$; the total degree will always be at most $r + 1 + (n - r - 2) + 1 = n$.

This proves that

$$\begin{aligned} \mathbb{P}_{n,\mathbb{R}} &\supsetneq \mathcal{L}_{A_{0,0}} \supsetneq \mathcal{L}_{A_{0,0},A_{0,1}} \supsetneq \cdots \mathcal{L}_{A_{0,0},\ldots,A_{0,n-1}} \\ &\supsetneq \cdots \mathcal{L}_{A_{0,0},\ldots,A_{n-1,0}} \supsetneq \mathcal{L}_{A_{0,0},\ldots,A_{n-1,1}} \supsetneq \mathcal{L}_{A_{0,0},\ldots,A_{n-1,1},P} \end{aligned}$$

If $P \in \cup_{i=0}^{n-1} \ell_i^1$, then $P \notin \cup_{i=0}^{n-1} \ell_i^2$ and we can repeat the above construction with the roles of $\{\ell_i^1\}$ and $\{\ell_i^2\}$ exchanged. So, in any case,

$$\dim\left(\mathcal{L}_{A_{0,0},\ldots,A_{n-1,1},P}\right) = \dim(\mathbb{P}_{n,\mathbb{R}}) - \#(I) - 1 = 0,$$

as required. □

**Note.** A similar argument shows that the points,

$$\alpha = (\alpha_0, \ldots, \alpha_k) \in \mathbb{N}^{k+1} \text{ for } |\alpha| = \alpha_0 + \cdots \alpha_k = n,$$

impose linearly independent conditions on the vanishing of homogeneous polynomials of degree $n$ in $k+1$ variables. This was conjectured in [**3**].

**Theorem.** *Let $\ell$, $\ell_0^1$, $\ell_0^2 \in \check{\mathbb{P}}_{\mathbb{R}}^2$ be lines in general position. Let $\ell_1^1, \ldots, \ell_{n-1}^1$ be close enough to $\ell_0^1$ and $\ell_1^2, \ldots, \ell_{n-1}^2$ close enough to $\ell_0^2$, so that the two sets of points $\{B_h^1 = \ell \cap \ell_h^1,\ h = 0, \ldots, n-1\}$ and $\{B_h^2 = \ell \cap \ell_h^2,\ h = 0, \ldots, n-1\}$ are not interlaced in $\ell$. Then, perhaps after moving the line $\ell$ slightly, there are $2(n-1)$ distinct non singular curves of degree $n$ through $A_{i,j} = \ell_i^1 \cap \ell_j^2$, $i + j \leq n$, $0 \leq i, j \leq n-1$ and tangent to $\ell$.*

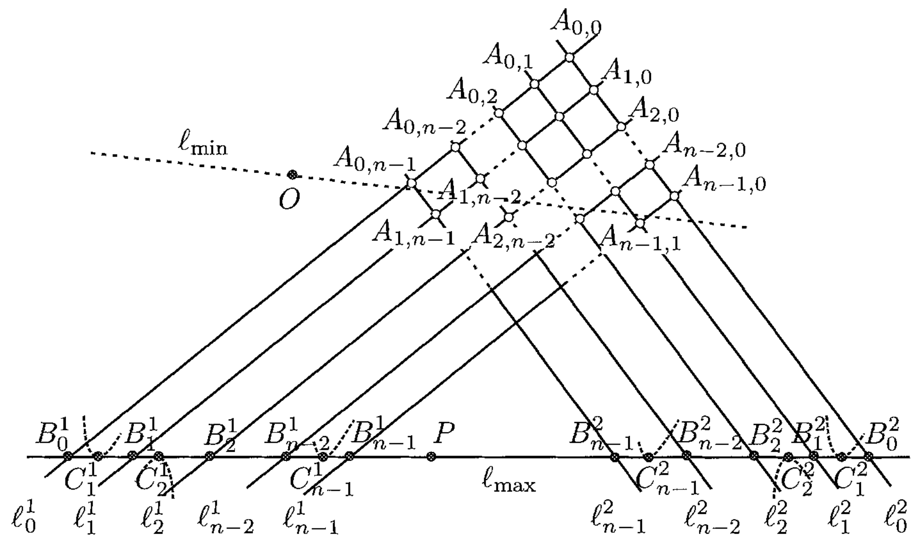


Figure 1

*More precisely, let $I_1,\ I_2$ be intervals such that*

$$B_0^1, \dots, B_{n-1}^1 \subset I_1, \text{ and } B_0^2, \dots, B_{n-1}^2 \subset I_2;$$

*choose orders on $I_1,\ I_2$, and assume $B_0^1 < \cdots < B_{n-1}^1$ and $B_0^2 < \cdots < B_{n-1}^2$. Then there exist points $C_h^1,\ C_h^2 \in \ell$, $h = 1, \dots n-1$, with $B_{h-1}^1 < C_h^1 < B_h^1$ and $B_{h-1}^2 < C_h^2 < B_h^2$, such that for each $B_h^1$ or $B_h^2$ there exists a curve of degree n through the $A_{i,j}$, $0 \le i, j \le n-1$, $i + j \le n$, tangent to $\ell$ at $B_h^1$ or $B_h^2$. Perhaps after moving the line $\ell$ slightly, these curves are non singular and distinct.*

**Proof.** Set $I = \{(i, j) \mid i + j \le n,\ 0 \le i, j \le n-1\}$. Consider the map,

$$\phi_\ell \colon \ell \to \mathcal{L}_{\{A_{i,j}\}_{(i,j)\in I}},\ \phi_\ell(P) = \text{the unique curve through P and the } A_{i,j}\text{'s for } (i,j) \in I.$$

Denote by $f_1$ the curve constituted by the union of $n$ lines, $\ell_0^1 \cup \cdots \cup \ell_{n-1}^1$, and by $f_2$ the curve constituted by $\ell_0^2 \cup \cdots \cup \ell_{n-1}^2$; then $f_1,\ f_2 \in \mathcal{L}_{\{A_{i,j}\}_{(i,j)\in I}}$ and

$$\phi_\ell^{-1}(f_1) = \left\{B_0^1, \dots, B_{n-1}^1\right\},\ \phi_\ell^{-1}(f_2) = \left\{B_0^2, \dots, B_{n-1}^2\right\}.$$

So we can apply Proposition 2 to infer that $\phi_\ell$ has $2(n-1)$ critical points $C_1^1, \dots, C_{n-1}^1, C_1^2, \dots, C_{n-1}^2$, with $B_{h-1}^1 < C_h^1 < B_h^1$ and $B_{h-1}^2 < C_h^2 < B_h^2$ for $h = 1, \dots, n-1$ (see Figure 1).

Among the curves through the $A_{i,j}$'s, $(i, j) \in I$, there is only a finite number of singular ones. Also, the space of lines of $\mathbb{P}^2_{\mathbb{K}}$ that are tangent at 2 or more distinct points of one of these curves is of dimension 1. It follows that, perhaps after moving the line $\ell$ slightly, $\phi_\ell$ has distinct critical values, corresponding to non singular curves. These are $2(n-1)$ distinct non singular curves through the $A_{i,j}$'s, $(i, j) \in I$, tangent to $\ell$. □

## 5. Configurations with an intermediate number of real solutions

**Proposition 6.** *There exist configurations of points $P_2, \dots, P_{N_n}$ and a line $\ell$ such that there are exactly $2(n-1)-2r$ curves of degree $n$ through $P_2, \dots, P_{N_n}$, tangent to $\ell$.*

**Proof.** The case $r = 0$ was done in the previous section. For $r = n-1$, we can take the line $\ell_{\min}$ shown in Figure 1; the points of intersection of $\ell_{\min}$ with the two sets of lines $\ell^1_0, \dots, \ell^1_{n-1}$ and $\ell^2_0, \dots, \ell^2_{n-1}$ are alternate, and it follows from Proposition 3 that the corresponding morphism $\phi_{\ell_{\min}} : \ell_{\min} \to \mathcal{L}_{\{A_{i,j}\}_{(i,j)\in I}}$ has no real ramification points.

There remain the cases $1 \le r \le n-2$; for these, we must invoke corollary 2 of Proposition 4. Take the affine line $\ell = \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}$ and $\alpha_1, \dots\dots, \alpha_n, \beta_1, \dots, \beta_n \in \ell$ such that the corresponding morphism,

$$\Theta(\alpha, \beta) = \left[\prod_{i=1,\dots,n} (x - \alpha_i t), \prod_{i=1,\dots,n} (x - \beta_i t)\right] : \mathbb{PR}^1 \to \mathbb{PR}^1$$

has $2(n-1)-2r$ real ramification points exactly. Choose lines in general position $\ell^1_0, \dots, \ell^1_{n-1}, \ell^2_0, \dots, \ell^2_{n-1}$, with $\ell^1_i \cap \ell = \alpha_{i+1}, \ell^2_i \cap \ell = \beta_{i+1}, i = 0, \dots, n-1$, and set $A_{i,j} = \ell^1_i \cap \ell^2_j, i, +j \le n, 0 \le i, j \le n-1$. Then the corresponding morphism $\phi_\ell : \ell \to \mathcal{L}_{\{A_{i,j}\}_{(i,j)\in I}}$ has exactly $2(n-1)-2r$ real ramification points and we can argue as in the proof of the theorem of § 3 to conclude. □

**Remark.** The mutual positions of the $\alpha_i$'s and $\beta_i$'s determine a lower bound for the number of real ramification points. Namely, working in $\mathbb{PR}^1$ rather than in $\mathbb{R}$, say

$$\mathbb{PR}^1 \setminus \{\beta_1, \dots, \beta_n\} = I_1 \cup \dots \cup I_n, \text{ and } \mathbb{PR}^1 \setminus \{\alpha_1, \dots, \alpha_n\} = J_1 \cup \dots \cup J_n$$

where the $I_h$'s are disjoint intervals, as well as the $J_h$'s. Then, if

$$m_h = \#(I_h \cap \{\alpha_1, \dots, \alpha_n\}),\ n_h = \#(J_h \cap \{\beta_1, \dots, \beta_n\}) \text{ for } h = 1, \dots, n,$$

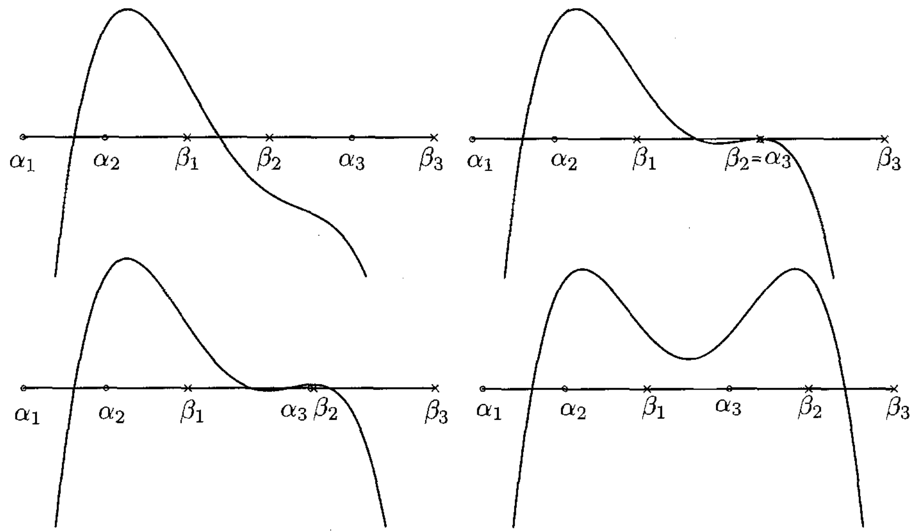


Figure 2

it follows from Proposition 2, or directly from Rolle's theorem, that the number of ramification points is at least,

$$\sigma_{\min} = \sum_{m_h \geq 1} (m_h - 1) + \sum_{n_h \geq 1} (n_h - 1). \tag{$\heartsuit$}$$

Only in the two extreme cases (the corollary of Propositions 2, and Proposition 3), can we deduce the exact number of real ramification points from the mutual positions of the $\alpha_i$'s and $\beta_i$'s.

In fact, it is easy to produce explicitly all possible values for the Formula ($\heartsuit$) in the situation of Proposition 5. On Figure 1, we start with the line $\ell_{\min}$ and let it rotate with center at the point $O$. Then the line will cross successively the points $A_{n-1,1}$, $A_{n-2,2}, \ldots, A_{1,n-1}$ and at each crossing the number $\sigma_{\min}$ increases by 2. However, it is not at all clear how the number of real solutions to our enumerative problem will vary. Figure 2 shows an example with

$$\alpha_1 < \alpha_2 < \beta_1 < \beta_2 < \alpha_3 < \beta_3;$$

the $\alpha$'s are represented by circles, the $\beta$'s by crosses and the graph represents $\varphi(x, \alpha, \beta)$. Then $\alpha_3$ decreases, coincides with $\beta_2$, then takes its place between $\beta_1$ and $\beta_2$. When $\alpha_3 = \beta_2$, the graph is simply tangent to the x-axis, in accordance with Proposition 4. But when it decreases still a little bit, 2 additional ramification points (or zeroes of $\varphi$) appear between $\beta_1$ and $\alpha_3$; they disappear when the $\alpha$'s and $\beta$'s are distributed more evenly.

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